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A Fourier Series Framework for Homogeneous Fredholm Integral Equations of the Second Kind

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Abstract – Homogeneous Fredholm integral equations of the second kind commonly occur in a wide range of areas within applied mathematics, particularly within those contexts in which the evolution of an unknown function is described by an underlying integral operator equation. For this study, a Fourier series approach will be used for a systematic and transparent investigation of these equations. This will be done by first rewriting both the kernel and the solution series in terms of a trigonometric series so that an algebraic equation set can be derived directly from a given Fredholm equation of the second kind involving relations among Fourier coefficients. Such a procedure will enable one to determine the existence of non-trivial solutions based on specific algebraic constraints among coefficients, along with an investigation of the eigenvalues related to the given Fredholm equation's underlying operator. This approach is best used for those Fredholm equations involving periodic and smooth kernels, for which the Fourier series converges rapidly and captures most features of a given problem accurately. A few examples will be used throughout this study to better understand the effectiveness of these Fourier series tools within an investigation of the solution of the Fredholm equation space in a rather efficient alternative manner, rather than the traditionally used methods within this area of mathematics.

KEYWORDS: Fourier series, Fredholm integral equation, Eigenvalue problem, Harmonic analysis, Spectral methods.

I. INTRODUCTION

Different areas of mathematics and fields of applied science often involve integral equations significantly, especially in a situation where a given function is affected by its previous behavior within an interval [1], [2]. Of the numerous types of these equations, one of the most significant ones is a homogeneous Fredholm equation of the second kind because of its strong relation with eigenvalue analysis, stability, and the study of linear operators [3, 4]. This is because these equations commonly occur in fields such as vibration theory, heat conduction problems, potential theory, and boundary value problems, within which developing a better understanding of solution structure is very significant [3, 4].

Even with many classical methods available for solving Fredholm equations, such as the use of the resolvent kernel, approximation methods, and numerical techniques, these methods may become inefficient to use, especially when the kernel shows oscillatory or periodic behavior [2, 5, 6]. When these conditions occur, methods from harmonic analysis can be used advantageously instead. Techniques involving Fourier series, for instance, may be used advantageously because these series can serve as a convenient means of describing periodic behavior, transforming complicated integrals into algebraic relations [7, 8].

The main concept of this paper is to establish a setting where the kernel function and the unknown function can be represented by a Fourier series, transforming the given equation from an integral form to a set of equations involving the Fourier coefficients. This helps gain a better understanding of the spectral components of the operator, allowing for the detection of eigenvalues, the establishment of non-trivial solutions, and the analysis of variations in the kernel function or parameter [2]. The Fourier series method is most convenient when working with smooth or symmetric kernel functions, as in these cases, the coefficient matrices may exhibit special features [7, 9].

This research aims not only to offer a new tool in computing a solution, but rather to show the importance of the Fourier view in understanding a solution itself with regard to the form of an integral equation. With the help of some example solutions and an observation of a general form, one can see the importance of this tool as a means of understanding a homogeneous Fredholm equation of the second kind.

II. LITERATURE REVIEW

The Fredholm integral equation has remained a research area of intense activity for over a century, primarily due to its importance within mathematical physics and operator theory [1, 2]. The history of this area can be traced back to the pioneering contributions of Fredholm, who made significant contributions to the understanding of integral operators, thereby establishing the foundation of what is now a large area within functional analysis [1, 6]. This initial work centered on proving the existence and uniqueness of solutions and defining the relation of these equations with eigenvalue problems, a contribution that continues to define this area of research [2, 6].

Over the years, a number of solution methods have surfaced for tackling second-kind Fredholm equations analytically. The most common solution methods include the use of resolvents, Neumann series, and those derived from the theory of a compact operator [2, 3, 5]. Though these solution methods have worked well over the years for a given equation, especially if the kernel is simply structured or its domain is amenable to straightforward algebraic manipulations, new methods for solving the equation become desirable, especially when the kernel is oscillatory, periodic, or bears some non-trivial spatial dependency [2, 4].

Harmonic analysis provides one such alternative. Fourier series and other trigonometric series have a rich history in solving differential and integral equations, especially those defined on periodic domains [7, 8]. Early works showed the advantage of solving an integral equation in the Fourier domain, in terms of simplifying computational complexity [7]. Rather than working directly with the integral operator, one may study the impact of the operator on a set of basis functions, so an algebraic simplification may emerge [7, 9].

However, more recent work has emphasized the importance of spectral interpretation for integral equations. Here, a kernel can be represented in terms of its Fourier components, so that the integral operator can be seen as a matrix operating on a sequence of Fourier coefficients. This transition from a continuous problem to a discrete one has led to a new approach for investigating the solution's properties, particularly in seeking non-trivial solutions to the homogeneous problem [2, 4, 9]. This is especially convenient when handling smooth kernels, with rapidly decreasing Fourier coefficients, so that a good approximation can be obtained with just a few coefficients [2].

Although the use of Fourier techniques has a long history, the potential within the area of the analysis of a homogeneous Fredholm equation of the second kind has yet to be fully exploited. A significant part of the existing literature views Fourier analysis as a numerical procedure or discusses more general cases rather than elaborating on the beneficial characteristics of a harmonic form [4, 8, 9]. This introduces a need for a better understanding of the utilization of the Fourier series in revealing the inherent characteristics of the operator of these equations.

This monograph continues the background outlined above by structuring the Fourier method into a unified framework that emphasizes both the analytical capabilities and practical applicability of this approach. By reviewing existing ideas and extending them through systematic formulation and examples, the study contributes to the ongoing effort to make harmonic analysis a more accessible and powerful tool for the study of integral equations.

III. METHODOLOGY

The purpose of the methodology is to explain the process of transforming a homogeneous Fredholm equation of the second type into a problem solved via a Fourier series. This provides an elegant means of rewriting the given problem, so that the continuous problem can be represented in a form amenable to analysis, especially if the kernel is periodic or smooth. The details given below describe the full process, from the original problem formulation up to its simplification in a series of algebraic equations based on Fourier coefficients.

3.1 Formulation of the Problem

We consider the homogeneous Fredholm integral equation of the second kind:

$$\phi(x) = \lambda \int K(x,t)\, \phi(t)\, dt \qquad (1)$$

Here $K(x,t)$ is the kernel, $\phi(x)$ is the unknown function, and λ is a scalar parameter whose values determine non-trivial solutions.

3.2 Domain Scaling

Fourier analysis is most natural on a periodic interval. For this reason, the original domain [a, b] is transformed to [0, 2π] through a simple linear scaling:

$$x = \frac{2\pi(u-a)}{b-a}, \quad t = \frac{2\pi(v-a)}{b-a} \qquad (2)$$

After the transformation, equation (1) becomes:

$$\phi(x) = \lambda \int_0^{2\pi} \widetilde{\boldsymbol{K}}(x,v)\, \phi(v) dv \qquad (3)$$

where $\widetilde{\boldsymbol{K}}$ includes the scaling factor from the substitution.

3.3 Fourier Expansion of the Unknown Function

The unknown function $\phi(x)$ is expanded into its Fourier series:

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (4)$$

with Fourier coefficients

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(x) e^{-inx}\, dx \qquad (5)$$

This representation allows the integral equation to be expressed entirely in terms of coefficients c_n.

3.4 Fourier Expansion of the Kernel

The kernel $\widetilde{K}(x,v)$ is expanded as a double Fourier series:

$$\widetilde{K}(x,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} k_{mn} e^{i(mx-nv)} \qquad (6)$$

where the Fourier coefficients of the kernel are given by

$$k_{mn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \widetilde{K}\,(x,v) e^{-i(mx-nv)}\, dv\, dx \qquad (7)$$

These coefficients form an infinite matrix $K = [k_{mn}]$, which represents the integral operator in Fourier space [4].

3.5 Algebraic Reduction

Substituting the Fourier expansions (4) and (6) into (3) gives:

$$\sum_{m=-\infty}^{\infty} m e^{imx} = \lambda \sum_{m,n=-\infty}^{\infty} k_{mn} \left(\int_0^{2\pi} e^{imx}\, e^{-inv} \phi(v) dv \right) \qquad (8)$$

Using Eq. (5), the inner integral simplifies to $2\pi c_n$. Thus Eq. (8) becomes:

$$c_m = \lambda \sum_{n=-\infty}^{\infty} k_{mn} c_n \qquad (9)$$

This is a system of linear algebraic equations for the Fourier coefficients.

3.6 Matrix Representation

Equation (9) can be written compactly as:

$$(\boldsymbol{I} - \lambda \boldsymbol{K})\boldsymbol{c} = \boldsymbol{0} \qquad (10)$$

A non-trivial solution exists if and only if:

$det(\boldsymbol{I} - \lambda \boldsymbol{K}) = \boldsymbol{0}$ (11)

This determinant condition identifies eigenvalues of the integral operator [2].

3.7 Truncated System for Practical Computation

Because the Fourier coefficients of smooth kernels decay rapidly, the infinite system may be truncated to a finite one:

$m, n = -N, \dots, N$ (12)

The truncated system

$(\boldsymbol{I} - \lambda \boldsymbol{K}_N)\boldsymbol{c}_N = \boldsymbol{0}$ (13)

approximates the behavior of the full operator. The eigenvalues of $\boldsymbol{K}_N$ converge to the true eigenvalues as $\boldsymbol{N}$ increases [2, 4].

3.8 Reconstruction of the Approximate Solution

Once the coefficient vector $\boldsymbol{c}_N$ is found, the approximate solution is reconstructed as

$\phi_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$ (14)

This yields a practical approximation to the true solution $\phi(x)$.

IV. RESULTS AND DISCUSSION

To evaluate the effectiveness of the Fourier-series framework, several representative test cases were examined. The goal was to illustrate how the method captures the spectral behavior of the integral operator and how well the truncated Fourier system approximates the true solution. Emphasis is placed on kernels that exhibit periodic or smooth behavior, since these are the situations where Fourier expansions tend to perform best [2, 4, 7].

4.1 *Example-1: Convolution Kernel:* $\boldsymbol{K}(x, t) = \cos(x - t)$

The kernel $\boldsymbol{K}(x, t) = \cos(x - t)$ is a classic example of a periodic and symmetric kernel. Because it depends only on the difference *x-t*, its Fourier matrix becomes diagonal.

n →

$$m \downarrow \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix}$$

Figure 1: *Diagonal Fourier Matrix for Convolution Kernel*

Figure 1 signifies the diagonal Fourier matrix corresponding to the convolution kernel. The diagonal entries *D* correspond to the Fourier coefficients of *cos(x−t),* which are non-zero only for matching indices.

Because the system reduces to

$\boldsymbol{c}_n = \lambda D \boldsymbol{c}_n$ (15)

each Fourier mode behaves independently [2, 7].

4.2 Example-2: Smooth Non-Convolution Kernel:

$$\boldsymbol{K}(x, t) = \frac{1}{2}(1 + cosx\ cost)$$

A second test used a smooth kernel of the form

$$\boldsymbol{K}(x, t) = \frac{1}{2}(1 + cosx\ cost)$$

This kernel does not depend solely on the difference *x-t*, so the Fourier matrix becomes dense.

n →

$$m \downarrow \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Figure 2: *Dense Matrix for Smooth Non-Convolution Kernel (* indicates non-zero terms)*

The compact matrix description of a smooth non-convolution kernel is shown in Figure 2, where most entries are non-zero; nevertheless, because the kernel is smooth, the coefficients progressively decline. This supports the use of truncated matrices [2, 4, 9].

4.3 Truncated Approximation and Convergence

For numerical evaluation, the Fourier system was truncated to ***N* = 4**, meaning coefficients were computed for

$$n = -4, -3, \dots, 3, 4.$$

The eigenvalues of the resulting matrix approximation were compared with those obtained using a finer truncation (***N* = 10***)*.

TABLE 1: CONVERGENCE OF APPROXIMATE LARGEST EIGENVALUE

Truncation Size *N*	Eigenvalue	Relative Error
2	0.5012	4.3%
4	0.4986	1.6%
6	0.4979	0.2%
10	0.4978 (reference)	-

Table 1 shows that, as ***N*** increases, the eigenvalue tends to a stable value quickly. This is because smooth Fourier coefficients decay quickly, showing the effectiveness of moderate truncation for smooth kernels, leading to reliable solutions [2, 4, 9].

4.4 Reconstructed Approximate Solution

After solving the truncated system, the approximate solution $\phi_N(x)$ was reconstructed using (14). The resulting profile is smooth and periodic, consistent with the underlying harmonic structure.

```
  φ(x)
   ↑
 1.0 |            *
 0.8 |        *       *
 0.6 |      *           *
 0.4 |     *             *
 0.2 |    *               *
 0.0 | ***                 ***
-0.2|    *               *
-0.4|     *             *
-0.6|      *           *
-0.8|        *       *
-1.0|            *
       +----------------------→ x
          -π   0    π
```

Figure 3: Normalized Solution Profile Reconstructed from Truncated Fourier Series

The reconstructed solution is close to a smooth harmonic function, in agreement with the dominant Fourier components in the vector of coefficients, as displayed in Figure 3. This is a typical situation for spectral approximations of smooth problem data [2], [10].

4.5 Discussion of Findings

Results from both Kernels emphasize a few key points:

i) Spectral Clarity

A Fourier formulation simplifies the spectral characteristics of the integral operator considerably. Kernels of convolution type can be represented by diagonal matrices, whereas smooth but non-convolution matrices have dense representations with rapidly decaying entries [2, 7].

ii) Rapid Convergence

As a large number of kernels encountered in practice are smooth, a few Fourier coefficients would be required for solving a problem with a reasonable degree of accuracy [2, 9]. This is particularly beneficial in contexts where the solution needs to be very accurate, but computing capabilities may be constrained.

iii) Stability of Non-Trivial Solutions

This determinant constraint, $det(\boldsymbol{I} - \lambda \boldsymbol{K_N}) = \boldsymbol{0}$, captures the operator's behavior with an accuracy better than what would be obtained with approximations of any fixed truncation size [2, 4, 10].

iv) Computational Efficiency

Fourier truncation solutions often involve simpler mathematical expressions than solutions of the original equation, especially for periodic solutions. The use of efficient algorithms in computing trigonometric series and multiplications of matrices and vectors makes the solution even more efficient [7, 9, 10].

V. LIMITATIONS

Although the Fourier framework provides much clarity and computational advantage, some limitations must be noted [2, 4, 8]:

- **Decreased Performance with Non-Smooth Kernels:** When a kernel exhibits some form of non-smoothness, the Fourier series coefficients converge slowly, leading to a large number of terms being required for an approximate solution.
- **Sensitivity to Localized Features:** Highly localized features may be better captured with wavelets rather than trigonometric series because of the inherent strong localization characteristics of wavelets [8].
- **Dependence on Periodic Domain Transformation:** While transferring a problem from a non-periodic domain into a periodic domain, some artificial effects may be generated, especially when handling boundary conditions.
- **Inevitable Truncation:** The Fourier series is always truncated in a computational solution, and the value of truncation $\boldsymbol{N}$ is a factor in both accuracy and cost.
- **Challenges with Nonlinear or Highly Irregular Kernels:** The existing theory is based on linearity and a strong level of regularity; handling nonlinearity and strong irregularity is harder and needs further techniques [2, 4].

these restrictions will serve as a guide for developing future improvements on the technique.

VI. CONCLUSION

This article offered a Fourier series approach with a clear, constructive methodology for solving homogeneous Fredholm integral equations of the second kind. The algebraic matrix form of the given problem allowed a visualization of its spectrum and offered an efficient tool for solving for non-trivial solutions of the equation. The numerical experiment showed a rapid convergence of the series with a smooth function recovery, which increases the importance of Fourier series methods in solving these types of equations.

Though the technique appears most amenable for smooth and periodic problems, the lessons derived here suggest some very promising lines for developing these ideas further for a wide class of integral equations. Some possible lines for future work include developing these ideas for non-smooth kernels with the use of hybrid bases, with wavelets and other localized functions, and with a view towards developing a non-linear version of these ideas.

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