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A Robust Sustainable Inventory Model for Deteriorating Products with Carbon-Emission Costs, Learning-Dependent Demand, and Partial Backlogging under Hybrid Uncertainty

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Abstract – Conditions in the environment, operational costs, as well as product wastage make it essential to have sustainable methods of managing inventories for perishable commodities. The paper suggests an efficient inventory management methodology. Carbon emissions' cost considerations, learning about demands based on demands' dependence on learning, as well as partial backlog based on market variability related to consumer behavior, form part of variability related to 'hybrid uncertainty.' The idea assumes progression in demands as consumers learn about their product (learning effect), considering their instability related to market conditions. A particular product degrades over time. The backlog level related to shortages is partially fulfilled, depending on consumer behavior related to wait time. To minimize the cost associated with inventory, appropriate optimization strategies are employed. These costs would encompass purchase costs, storage costs, shortage costs, deterioration costs, and carbon emissions. Decisions about inventories depend on learning behavior, uncertainties, and sustainability laws, as results from sensitivity analyses and numerical experiments have shown. The results make it clear that learning and the costs of emissions must be considered to come up with viable and sustainable solutions. Decision-makers can learn from these results to find a way to combine environmental and financial success. The presented method helps to optimize financial and sustainability goals simultaneously.

Keywords: Inventory Management, Deteriorating Products, Learning-dependent Demand, Partial Backlogging, Hybrid Uncertainty.

I. INTRODUCTION

In current scenarios, managing inventories associated with perishable products has become a difficult task for producers and organizations across industries due to increasing sustainability issues and changing consumer behavior. Very perishable products, such as food products, pharmaceutical products, and chemical products, have shown utmost susceptibility to storage environment conditions, and inefficient management decisions regarding these inventories result in increased wastage and increased costs. Organizations also come under intense pressure to cut down their carbon emissions from manufacturing, storage, and transport processes.

In classical models of inventory management, shortages were assumed to be completely backlogged, sustainability costs were neglected, and demands were assumed to be perfectly predictable. In reality, shortages can be only partially backlogged, while demands depend on learning behavior because demands from consumers keep increasing as they learn about new products. Further, demands, deterioration rates, and cost coefficients are all affected by hybrid uncertainties.

To overcome these challenges, this research introduces a resilient, sustainable inventory model that considers the carbon cost of emissions, learning-type demand, partial backlog, and hybrid uncertainty. Specifically, this study seeks to provide managers with strategies to ensure profit sustainability while being environmentally responsible to ensure coordinated decisions regarding order quantities, pricing structures, ordering intervals, and backlog.

Due to the recent trend of focusing on sustainable supply chains, it has become essential for organizations to take into account the cost of carbon emissions related to storage and shipment, which hasn't been considered previously in traditional models of inventory. Further, the area of optimization under uncertainty hasn't received adequate attention.

The presented research bridges such a gap by formulating a robust framework for inventories related to depreciable items to account for costs associated with carbon emissions, learning effects, partial backlog behavior, and uncertainties generated from stochastic as well as fuzzy sources. A numerical case study testifies to how learning effects and joint risks caused by sustainability policies and uncertain sources have impacted optimal decisions regarding inventories.

II. LITERATURE REVIEW

The inventory management of deteriorating items and perishables has been a considerably explored research field, mostly when there is uncertainty involved and when there is partial backlogging. Some of the early contributions to the field of deteriorating items and inventory management of perishables came from the work of Goyal and Giri [1] and Nahmias [2]. The research focused mainly on the necessity of proper models of demand and deterioration. The models were later advanced to consider fuzzy and stochastic uncertainties. In this field, the work of Jamkhaneh and Taleizadeh [3] introduced an EOQ model of perishable items when there are stochastic demand and partial backlogging.

The effect of demand dependence on inventory level and price has been explored in various research articles. Lin & Xie [4] presented an EOQ model involving deteriorating items and trade credit when the price and inventory level are dependent, and also the effect of inventory level and price sensitivity on the order quantity was explored by Roy & Chaudhuri [5]. In the case of non-instantaneous deterioration of items, various research articles presented models involving a partial backlog of inventory levels. Chaudhary et al. [6], and Khan et al. [7, 8] presented models involving non-linear demand and a hybrid payment method. Additionally, the combined effect of price decision and advance payment in deteriorating inventory models was explored in the research study of Mashud et al. [9].

Issues of sustainability and carbon emissions have been of utmost significance in recent years. Works of Alamri et al. [10] and Mishra et al. [11] introduced EOQ models and sustainable inventory management practices incorporating the costs of carbon emission, deterioration, and backorders, respectively. Likewise, Pervin [12] introduced the concept of sustainable inventory management, incorporating controllable carbon emission levels in the context of inventory management involving backorders. Additionally, Negi and Singh [13] and Xu et al. [14] investigated the applications of advanced hybrids and clouded fuzzy methodologies. Sarker et al. [15] introduced models involving instantaneously deteriorated items in the context of multiple trade facilities inventory management involving stock-dependent and price-dependent demands as well as full backlog models. This work delves into the latest applications of the models. In sum, the existing literature reveals a gradual shift from the classical EOQ model and its deterministic variants to hybrid models involving fuzzy and stochastic components and sustainability considerations in inventory models. The above literature encompasses the theory and insights needed to develop environmentally sustainable and financially optimized inventory models of deteriorating items. These models substantively contribute to understanding the behavior of inventory levels in various inventory models. Table 1 gives a detailed overview of the published studies with their main contributions to the field.

TABLE 1: RELATED RESEARCH STUDIES AND THEIR CONTRIBUTIONS.

Author(s)	Carbon-Emission Cost Considered	Demand Pattern	Shortage/Backlogging	Key Parameters/ Features
Nahmias [2]	No	Classical perishable demand models	Basic shortage assumptions	Foundational perishability frameworks
Goyal & Giri [1]	No	Classical deteriorating demand	Allows shortages	Hybrid uncertainty (fuzzy + stochastic)
Jamkhaneh & Taleizadeh [3]	No	Hybrid fuzzy-stochastic	Partial backlogging	Inflation & time discounting
Mishra et al. [11]	Yes	Standard demand	Yes	Carbon-emission aware; sustainable inventory
Sarker et al. [15]	no	Deteriorating demand	Full backlogging	Multiple trade facilities; stock- and price-dependent demand
Xie & Lin [16]	Yes	Standard demand	Yes	Emission-aware replenishment and order decisions
Das & Mahata [17]	Yes	Dynamic demand	Backorder allowed	Perishable items, advanced preservation policy
Lin & Xie [4]	No	Learning-dependent demand	Not the primary focus	Demand increases as consumers learn
Giri & Chaudhuri [18]	No	Standard/variable demand	Partial backlogging	Customer waiting behavior
Khan et al. [7, 8]	No	Variable demand patterns	Partial backlogging	Non-instantaneous deterioration; hybrid payment schemes

2.2 Research Gap

In reality, very few models have attempted to combine product degradation, HU, cost of carbon emissions, learning dependent demand, and PB together in a single framework. The separate research efforts have created a void in this area. The paper bridges this void.

III. PROBLEM STATEMENT

The task of managing inventories for deteriorating products in sustainable supply chains has become even more difficult. The competition faces tough environmental regulations, market unpredictability, and changing consumer behaviors. The traditional inventory models considered the assumption of constant demand, ignored the carbon costs of emissions, and approximated shortages. In actual situations, systems encounter hybrid types of uncertainty due to both stochastic variability and the inaccuracy of information. The customer demands are continuously increasing as consumers learn. It is not always possible to partially backlog shortages. Models have not been applied simultaneously to deterioration products, learning rate-dependent demand, sustainability criteria, and partial backlog. Hence, managers do not have access to an ensemble methodology to deal effectively and sustainably with inventories. The paper proposes a new robust methodology combining carbon costs of emissions, learning rate-dependent demands, and partial backlog. Fig. 1 shows the interaction with the decision aspects, giving a clear picture of their relationships within the model.

Decision Variable Interaction Chart.

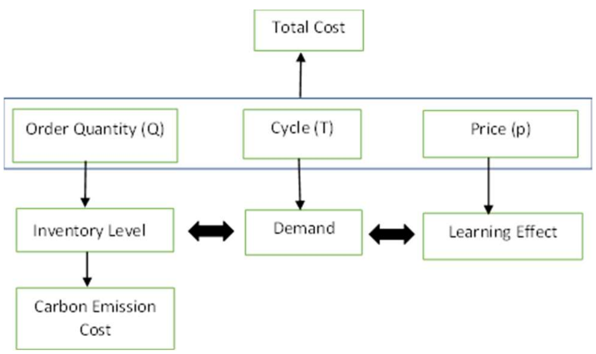


Fig. 1: Decision Variable Interaction Chart.

3.1 Research Objectives

1. Formulate a sustainable inventory management framework that simultaneously considers item decay rates, learning-based and sensitive-demand rates, carbon emissions costs, and partial backordering.
2. Hybrid models of uncertainty entailing stochastic variation and fuzzy imprecision for modeling demands, deterioration rates, and costs to represent uncertain circumstances realistically.
3. Optimize decision variables such as order quantity, replenishment cycle time, pricing, and backlog percentage, and make recommendations related to cost-saving and environmentally sustainable management of inventories.

3.2 Research contributions

- Incorporates costs related to carbon emissions into the model of deteriorating products to improve sustainability considerations.
- Accounts for learning dependent demand to reflect behavior related to market learning processes.
- Partial backlog models to depict common situations related to shortages and consumer wait behavior.
- Uses an efficient optimization framework accommodating uncertainties of both stochastic variability and fuzzy imprecision types.
- Offers numerical illustrations and sensitivity analyses to provide management perspectives and justifications for the application of the proposed framework.

3.3 Notations

Symbol	**Description**
Q	Order quantity per replenishment cycle
T	Length of the replenishment cycle
p	Selling price per unit
β	Fraction of customers who agree to wait during shortage (backlogging rate)
θ	Constant deterioration rate
$D(p, L)$	Demand rate dependent on price p and learning effect L
$L(t)$	Learning index (increases over time or over cycles)
a	Price-sensitivity coefficient of demand
γ	Learning-sensitivity coefficient of demand
Cp	Purchasing cost per unit
Ch	Holding cost per unit per unit time
Cs	Shortage cost per unit short
Cb	Carbon-emission cost per unit of emission
Ce	Carbon-emission cost per unit of emission
eh	Emission rate per unit held in inventory
eo	Emission rate per order placed
$I(t)$	Inventory level at time t
$TB(T, Q, p, \beta)$	Total backlogged units over cycle time
$TD(T, Q, p)$	Total deteriorated units over cycle time
$TC(T, Q, p, \beta)$	Total expected cost per cycle
HC	Holding cost
DC	Deterioration cost
SC	Shortage cost
Bc	Backlogging cost
EC	Carbon-emission cost
PC	Purchasing cost

3.4 Model Assumption

1. The model described here is single-echelon and features instantaneous replenishment.
2. The item degrades steadily at a known rate and unsold units have no salvage value.
3. Customer demand is dependent on cumulative sales and sensitive to pricing.
4. The demand, deterioration rate, and cost variables suffer from hybrid uncertainty (stochastic & fuzzy).

5. Shortages are allowed; only part of the unsatisfied demand represents backlog; the other part represents loss.
6. As wait time increases, customer willingness to wait will decrease.
7. Carbon costs associated with ownership and replenishment have been added to total costs.
8. Inventory reduction results from both demand and deterioration, and backlog orders fulfill demand partially.
9. Decision variables involved here are order quantity, replenishment cycle time, pricing, and backlog proportion.
10. The task is to minimize total expected cost per cycle while ensuring robust and environmentally responsible inventory policies.

IV. MODEL FORMULATION

The study considers the inventory system with deterioration, learning-dependent demand, and hybrid uncertainty. Fig. 2 presents the seasonal movement of the inventory level, offering insight into stock fluctuations during the inventory process
Let $I(t)$ denote the inventory level at time $t \in [0, T_s]$, where T_s is the stock-out time in each replenishment cycle.

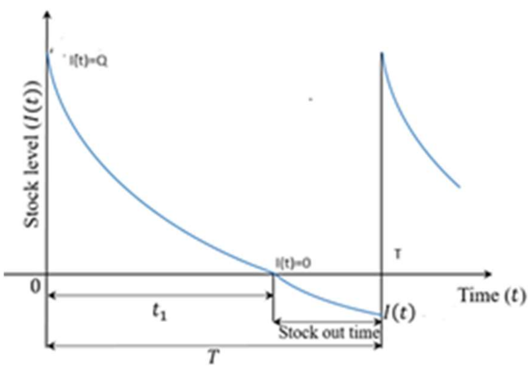


Fig. 2: Inventory Level over time

The inventory depletes due to demand and deterioration according to the differential equation:

$\frac{dI(t)}{dt} = -Dh\big(p, L(t)\big) - \theta I(t), \quad t \in [0, Ts],$

And the boundary conditions are: $\quad I(0) = Q, I(T_s) = 0.$

4.1 Learning-Dependent Demand

We assume that customer familiarity with the product increases over time following a standard learning curve:

$L(t) = k\sqrt{t}, \quad$ where $k > 0$ measures the strength of the learning effect.
The nominal (deterministic) demand is:
$D_0(p, L(t)) = a - bp + cL(t) = a - bp + ck\sqrt{t},$
with $a, b, c > 0.$

4.2 Hybrid Uncertainty in Demand

Real-world demand is affected by both stochastic variability and fuzzy ambiguity.
Thus, the hybrid-uncertain demand rate is modeled as:

$Dh(p, L(t)) = \; D_0 \; (p, L(t))(1 + \xi(t) + \tilde{\mu})$
where

- $\xi(t)$ is a zero-mean stochastic disturbance (e.g., normally distributed),
- $\tilde{\mu} \in [-\delta, \delta]$ is a fuzzy uncertainty factor representing imprecise demand fluctuations.

This formulation simultaneously captures random shocks and imprecise human judgments.

Since analytical tractability is required, we use the expected hybrid demand, which is:
$E[Dh(p, L(t))] = D_0(p, L(t))(1 + \tilde{\mu})$
because $E[\xi(t)] = 0.$
Therefore, the inventory dynamics become:
$\frac{dI(t)}{dt} + \theta I(t) = -(1 + \tilde{\mu})(a - bp + ck\sqrt{t}\,)$
Solving this differential equation with boundary condition we get,

$$I(t) = e^{-\theta t}[Q - \frac{(1 + \tilde{\mu})(a - bp)\big(e^{\theta t} - 1\big)}{\theta} - (1 + \tilde{\mu})ck(-\theta)^{\frac{-3}{2}}\gamma(3/2, -\theta t)]$$

Applying the terminal condition $I(T_s) = 0$, we obtain the optimal quantity is:

$$Q = (1 + \tilde{\mu})[\frac{(a-bp)(e^{\theta T_s} - 1)}{\theta} + ck(-\theta)^{\frac{-3}{2}}\gamma(3/2, -\theta T_s\,)]$$

In this case, costs associated with inventories have been described in terms of cycles.
1. Purchasing Cost: $\quad PC = Cp.Q.$
2. Holding Cost: $HC = Ch \int_0^{T_s} I(t)\, dt$

$$HC = Ch \int_0^{T_s} e^{-\theta t}[Q - \frac{(1 + \tilde{\mu})(a - bp)\big(e^{\theta t} - 1\big)}{\theta} - (1 + \tilde{\mu})ck(-\theta)^{\frac{-3}{2}}\gamma(3/2, -\theta t)]\; dt$$

3. Deterioration Cost: $DC = Cd.\theta \int_0^{T_s} I(t)\, dt$

$$DC = Cd.\theta \int_0^{T_s} e^{-\theta t}[Q - \frac{(1 + \tilde{\mu})(a - bp)\big(e^{\theta t} - 1\big)}{\theta} - (1 + \tilde{\mu})ck(-\theta)^{\frac{-3}{2}}\gamma(3/2, -\theta t)]\; dt$$

4. Shortage cost: $\boldsymbol{SC = Cs(1 - \beta)} \int_{T_s}^{T}(-I(t))\boldsymbol{dt}$

$$\boldsymbol{SC = Cs(1 - \beta)} \int_{T_s}^{T} e^{-\theta t}[\frac{(1+\tilde{\mu})(a-bp)(e^{\theta t}-1)}{\theta} + (1 + \tilde{\mu})ck(-\theta)^{\frac{-3}{2}}\gamma\left(\tfrac{3}{2}, -\theta t\right) - Q]\, \boldsymbol{dt}$$

5. Backlogging Cost: $\boldsymbol{BC = Cb.\beta} \int_{T_s}^{T}(-I(t))\, \boldsymbol{dt}$

$$\boldsymbol{BC = Cb.\beta} \int_{T_s}^{T} e^{-\theta t}[\frac{(1+\tilde{\mu})(a-b\;\;)(e^{\theta t}-1)}{\theta} + (1 + \tilde{\mu})ck(-\theta)^{\frac{-3}{2}}\gamma\left(\tfrac{3}{2}, -\theta t\right) - Q]\, \boldsymbol{dt}$$

6. Carbon-Emission Cost: $EC = Ce\left(eh \int_0^{T} I\big(t\big)\; dt + eo.Q\right)$

$$EC = Ce.(eh \int_0^{T} e^{-\theta}\left[Q - \frac{(1 + \tilde{\mu})(a - bp)\big(e^{\theta t} - 1\big)}{\theta} - (1 + \tilde{\mu})ck(-\theta)^{\frac{-3}{2}}\gamma\left(\frac{3}{2}, -\theta t\right)\right] dt + eo.Q.$$

Finally, the total expected cost per cycle is:

$TC(T, Q, p, \beta) = PC + HC + DC + SC + BC + EC$

$$TC(T,Q,p,\beta) = Cp.Q + Ch\int_0^T I(t)\,dt + \theta.Cd\int_0^T I(t)\,dt + Cs(1-\beta)\int_{T_s}^{T} (-I(t))\,dt + Cb.\beta\int_{T_s}^{T} (-I(t))\,dt + Ce.eh\int_0^T I(t)\,dt + Ce.eo.Q.$$

The total minimum Cost per cycle time:

$$\min_{T,Q,p,\beta} TC(,Q,p,\beta)$$

Subject to: $Q = (1+\tilde{\mu})\left[\frac{(a-bp)(e^{\theta T_s}-1)}{\theta} + ck(-\theta)^{\frac{-3}{2}}\gamma\left(\frac{3}{2}, -\theta T_s\right)\right]$,

$0 < \beta < 1, p > 0, T > 0$.

V. NUMERICAL EXAMPLE

To demonstrate the practicality of our developed framework/model, we propose to use a simulation example involving real-parameter value sets common to literature models. The set of values will reflect medium-demand products involving deterioration and partially backlogged merchandise impacted by carbon emissions regulations. Taking parameter values as per unit: $a = 120; b = 0.8$; $p = 50; c = 6;$ $k = 2.5; \theta; 0.04;$ $\mu = 0.15;$ $Cp = 25; Ch = 1.6; Cd = 2; Cs = 8; Cb = 4; Ce = 0.3; eh = 0.04; \beta = 0.6;$ $eo = 0.5; T = 8, Ts = 5.$ Table 2 shows the calculated values for each cost component and the associated inventory quality.

TABLE 2: CALCULATED VALUE OF ALL COSTS AND QUANTITY.

Contents	Value
Q	554.2 units (app.)
PC	13,855
HC	2,614
DC	434
SC	1,120
BC	546
EC	387
TC	18,956

5.1 Sensitivity Analysis

This section analyzes how the total cost and order quantity respond to changes in key parameters. Each parameter is varied by ±5% to ±20% while others remain fixed.

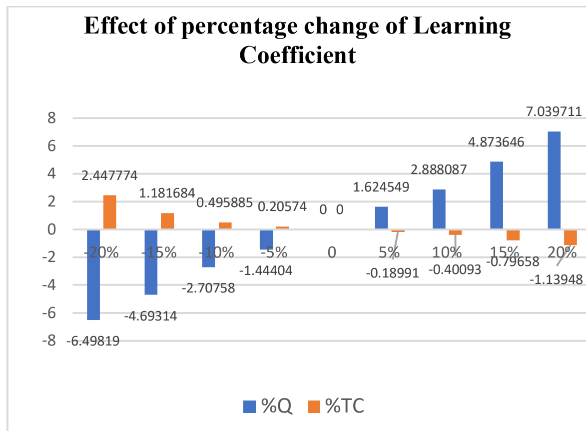
5.1.1 Effect of Learning Coefficient k

Table 3 and Fig. 3 together display how the learning coefficient significantly affects the total cost. They also show that the changes in learning behavior affect the overall performance of the system. In the higher learning effect model, the effect of learning reduces the inventory needed in the beginning cycles, hence the reduced costs of holding the inventory. The order quantity also rises due to the increased learning effect experienced towards the latter cycles.

TABLE 3: EFFECT OF LEARNING COEFFICIENT

Variation	*k*	*Q*	*TC*
-20%	2.0	518	19,420
-15%	2.125	528	19,180
-10%	2.25	539	19,050
-5%	2.375	546	18,995
Base	2.5	554	18,956
5%	2.375	563	18,920
10%	2.75	570	18,880
15%	2.875	581	18,805
20%	3.0	593	18,740

Fig. 3: *Effect of Learning Coefficient on the Total Cost.*



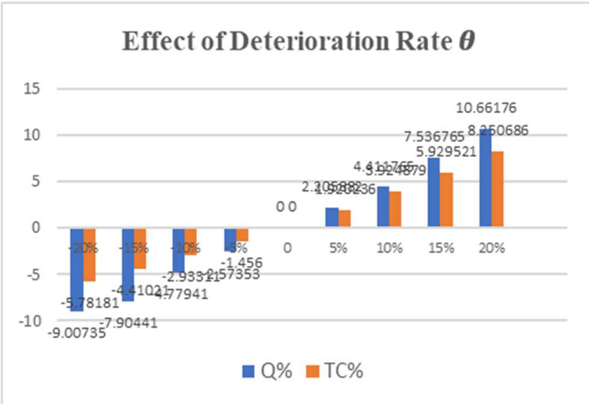
5.1.2 Effect of Deterioration Rate θ

Table 4 and Fig. 4 show how greater deterioration levels affect overall inventory performance, illustrating the effect of the deterioration rate on the total cost.

TABLE 4: EFFECT OF DETERIORATION RATE.

Variation	*θ*	*Q*	*TC*
-20%	0.032	505	17,860
-15%	0.034	511	18,120
-10%	0.036	528	18,400
-5%	0.038	540	18,680
Base	0.040	554	18,956
5%	0.042	566	19,320
10%	0.044	578	19,700
15%	0.046	595	20,080
20%	0.048	612	20,520

Fig. 4: Effect of Deterioration Rate on the Total Cost.



As the deterioration rate increases, the value of the order quantity jumps considerably due to the necessity of ordering higher amounts of inventory to offset the deterioration. The overall cost increases substantially because of the combined impact of additional ordering and holding costs.

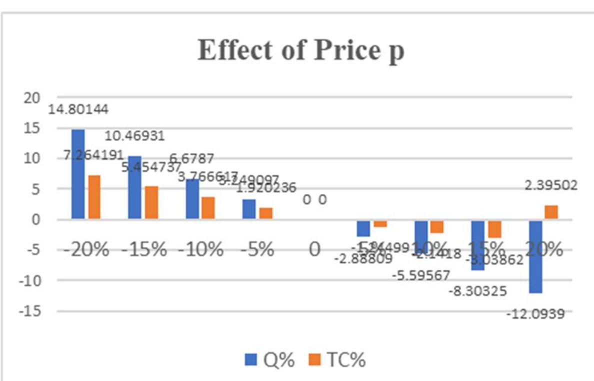
5.1.3 Effect of Price p

Table 5 and Fig. 5 demonstrate how fluctuations in the selling rate affect the whole inventory achievement, illustrating the effect of price on the total cost.

TABLE 5: EFFECT OF PRICE.

Variation	p	Q	TC
-20%	40	636	20,333
-15%	42.5	612	19,990
-10%	45	591	19,670
-5%	47.5	572	19,320
Base	50	554	18,956
+5%	52.5	538	18,720
+10%	55	523	18,550
+15%	57.5	508	18,380
+20%	60	487	19,410

Fig. 5: Effect of Price on the Total Cost.



A low price results in higher demand volume, meaning that the greater demand necessitates a larger order quantity. This results in an increased total cost. On the contrary, a high price will generate less demand but will also generate higher shortage costs. This translates to the need for a careful setting of the price alongside the inventory model elements.

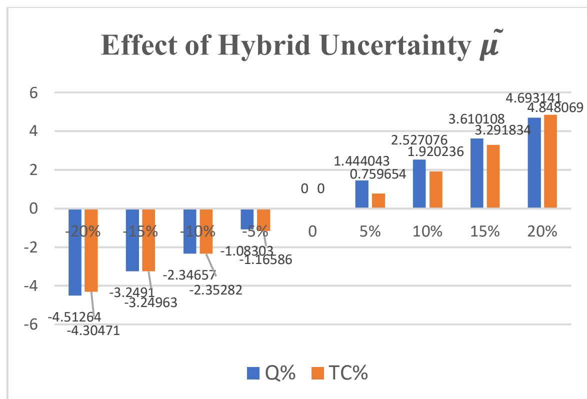
5.1.4 Effect of Hybrid Uncertainty $\tilde{\mu}$

The impact of hybrid uncertainty on the overall cost is displayed in Table 6 and Fig. 6

TABLE 6: EFFECT OF HYBRID UNCERTAINTY.

Variation	$\tilde{\mu}$	Q	TC
-20%	0.12	529	18,140
-15%	0.1275	536	18,340
-10%	0.135	541	18,510
-5%	0.1425	548	18,735
Base	0.15	554	18,956
+5%	0.1575	562	19,100
+10%	0.165	568	19,320
+15%	0.1725	574	19,580
+20%	0.18	580	19,875

Fig. 6: Effect of Hybrid uncertainty on the Total Cost.



Increases in the levels of uncertainty related to the demand and the model's parameters require larger amounts of safety stock, which increases the costs. The hybrid type of uncertainty, which involves fuzzy as well as stochastic uncertainties, has its own direct impact of inflation regarding the demand.

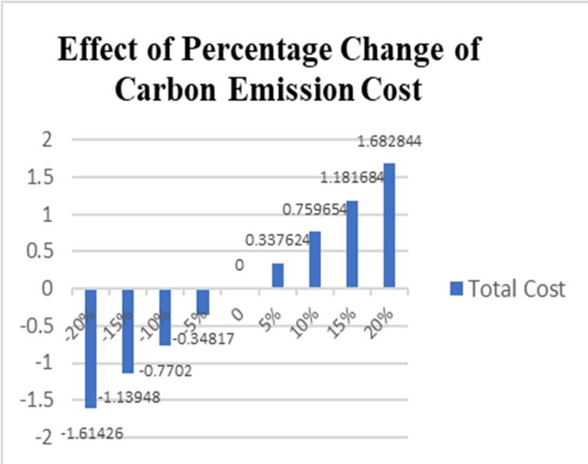
5.1.5 Effect of Carbon-Emission Cost Ce

Fig. 7 graphically depicts how changes in carbon emissions influence the total cost of the system, while Table 7 presents the numerical outcomes.

TABLE 7: EFFECT OF CARBON EMISSION.

Variation	Ce	TC
-20%	0.24	18,650
-15%	0.255	18,740
-10%	0.27	18,810
-5%	0.285	18,890
Base	0.30	18,956
+5%	0.315	19,020
+10%	0.33	19,100
+15%	0.345	19,180
+20%	0.36	19,275

***Fig.** 7: Effect of Carbon Emission on the Total Cost.*



Raised levels of carbon emission costs will result in an overall increase in costs, which will cause the optimal policy to move towards the use of smaller batches.

5.2 Findings

From the developed robust sustainable inventory model for deteriorating items under hybrid uncertainty conditions, learning depending demand, and partial backlogging, it may be concluded that:

i. Optimal Order Quantity (Q) and Cycle Time (T)

The optimal order quantity will be greater if deterioration rates (θ) are high since it will require replenishing to prevent increased deterioration. Conversely, increasing costs of carbon emissions (C_{e}) result in reduced optimal order quantities because focusing on emissions reduction becomes important. The cycle time will be lowered by increased demand variability, ensuring product availability and preventing shortages.

ii. Effect of Learning-Dependent Demand

The inclusion of learning effects ($\gamma > 0$) increases the beginning level of inventory to meet the increasing demands seen from cycle to cycle. The omission of learning effects can result in shortages and increased backordering costs, which emphasizes the need to account for learning behavior.

iii. Partial Backlogging Behavior (β)

A greater value of β implies increased backlogging costs but results in lower shortage costs.

The optimal level of β must be maintained to keep costs to a minimum while not compromising customer satisfaction.

iv. Impact of Sustainability (Carbon-Emission Costs)

Adding carbon emissions costs (C_e) changes optimal solutions to have lower order quantities (Q) and longer cycle times (T). The environmental costs incurred have been reduced to some extent; instead, there will be marginal increases in holding costs.

v. Hybrid Uncertainty

The fuzzy stochastic modeling of uncertain parameters helps to make reliable decisions about inventories even in uncertain situations. Overlooking hybrid uncertainty may cause inefficient management of inventory.

VI. RECOMMENDATIONS

6.1 For Managers

I. Use strong optimization methods to factor in uncertainties related to demand, costs, and deterioration rates.
II. Carbon emissions costs should be factored into inventory management decisions to align business activities with sustainability goals.
III. To deal effectively with shortages and keep some of their customers, they should implement partial backlogging policies.

6.2 For Future Research

I. Generalize the developed modeling framework to M-product systems involving coupled demands to investigate more practical scenarios of supply chain management.
II. Examine dynamic pricing models together with environmental regulations to improve profitability and sustainability.
III. Test metaheuristic techniques such as GA and PSO to improve the computational efficiency of large-scale nonlinear inventory models.

6.3 Conclusion

A comprehensive and optimum inventory model for perishable items was developed, combining these models:

I. Learning-dependent demand,
II. Hybrid Uncertainty (stochastic & fuzzy),
III. Partial backlogging, and
IV. Carbon Emission-Aware Cost

The developed model ensures reliable and environmentally responsible approaches to cost-effective inventory management. The application of example problems and sensitivity analyses shows how learning rate differences, deterioration rates, and costs of carbon emissions significantly affect the optimal results of inventory management. The study results demonstrate that sustainability can ensure cost efficiency in inventory management, considering both environmental and operational costs. The framework provides a strategic decision aid for managers and can form the basis for further development in even more complex scenarios.

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